$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Ex: $f(x) = x + e^x$. Find $(f^{-1})'(1)$.

Does f^{-1} exist? (Is f one-to-one)

If f increasing $\longrightarrow f'(x) > 0$

If f decreasing $\longrightarrow f'(x) < 0$

$f'(x) = 1 + e^x > 0 \longrightarrow f$ increasing $\rightarrow f$ one-to-one

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

$y = f(x)$
$\downarrow$
$f^{-1}(y) = x$

$f^{-1}(1) = 0$ $\quad$ $f'(0) = 1 + e^0 = 2$

$(x + e^x = 1)$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{2}$$

Implicit Differentiation

Ex: Find $\frac{dy}{dx}$ where y is implicity defined as a function of x by $x^2+y^2=1$.

$x^2+y^2=1 \longrightarrow \frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)$

$\hookrightarrow y^2=1-x^2$

$\hookrightarrow 2x+2y\cdot y'=0$

$\hookrightarrow y=\pm\sqrt{1-x^2}$

$\hookrightarrow y'=\frac{-x}{y}$

$y=\sqrt{1-x^2}$

$y'=\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)$

$=\frac{-x}{\sqrt{1-x^2}}$

$y'=\frac{-x}{y}$

$=\frac{-x}{\sqrt{1-x^2}}$

Ex: Find y' where $x^3 - xy^2 + y^3 = 1$.

$$\frac{d}{dx}\left(x^3 - \underbrace{xy^2}_{\text{Product rule}} + y^3\right) = \frac{d}{dx}(1)$$

$$3x^2 - \left(1 \cdot y^2 + x \cdot 2y\,y'\right) + 3y^2 \cdot y' = 0$$

$$3x^2 - y^2 - 2xyy' + 3y^2y' = 0$$

$$3x^2 - y^2 = 2xyy' - 3y^2y'$$

$$\hookrightarrow 3x^2 - y^2 = (2xy - 3y^2)y'$$

$$\hookrightarrow \boxed{y' = \frac{3x^2 - y^2}{2xy - 3y^2}}$$

Ex: If $y = \dfrac{e^{-x}\cos^2 x}{(x^2+2)^2\sqrt{x+1}}$, find y'.

Logarithmic Differentiation:

② $\ln(AB) = \ln A + \ln B$

① $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

③ $\ln(A^p) = p\ln(A)$

$$\ln y = \ln\left(\frac{e^{-x}\cos^2 x}{(x^2+2)^2\sqrt{x+1}}\right)$$

$$= \ln\left(e^{-x}\cos^2 x\right) - \ln\left((x^2+2)^2\sqrt{x+1}\right)$$

$$= \cancel{\ln}\left(\cancel{e}^{-x}\right) + \ln\left(\cos^2 x\right) - \left(\ln\left((x^2+2)^2\right) + \ln\left(\sqrt{x+1}\right)\right)$$

$$= -x + 2\ln(\cos x) - 2\ln(x^2+2) - \frac{1}{2}\ln(x+1)$$

$$\frac{d}{dx}\left[\ln y = -x + 2\ln(\cos x) - 2\ln(x^2+2) - \frac{1}{2}\ln(x+1)\right]$$

$$\frac{1}{y}y' = -1 - 2\tan x - 2\frac{2x}{x^2+2} - \frac{1}{2}\frac{1}{x+1}$$

$$y' = y\left(-1 - 2\tan x - \frac{4x}{x^2+2} - \frac{1}{2(x+1)}\right)$$

$$\boxed{y' = \frac{e^{-x}\cos^2 x}{(x^2+2)^2\sqrt{x+1}}\left(-1-2\tan x-\frac{4x}{x^2+2}-\frac{1}{2(x+1)}\right)}$$